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Lacunarity calculation in the true fractal limit

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Abstract. We measure the lacunarity of regular fractals in the limit of an infinite fractal lattice using a suitable graph counting method. The calculations are performed for a recent expression proposed for lacunarity that explores the scaling properties of the mass distribution of the fractal set on cells of radius r. We present results for various Sierpinski carpets and for the Sierpinski gasket.

In the pioneering work of Gefen *et al* [1], they found that, in contrast to Euclidean lattices, the critical properties of physical systems on fractal lattices [2] depend not only on the fractal dimensionality (D_F) but also on other geometrical factors such as lacunarity (Λ).

The possibility of characterizing the universality classes of these systems by such geometrical factors naturally arises, but for that it would also be necessary to introduce a mathematical definition of lacunarity which is intended to measure the extent of the failure of a fractal to be translationally invariant, or the degree of inhomogeneity of a fractal [2].

Since then, much effort have been made [1, 3–6] to define lacunarity in a precise manner. In general, the proposals were computed for the Sierpinski carpets, a particular family of fractals in which, from a basic square formed by $b \times b$ subsquares, m subsquares are cut out recursively (see figure 1).

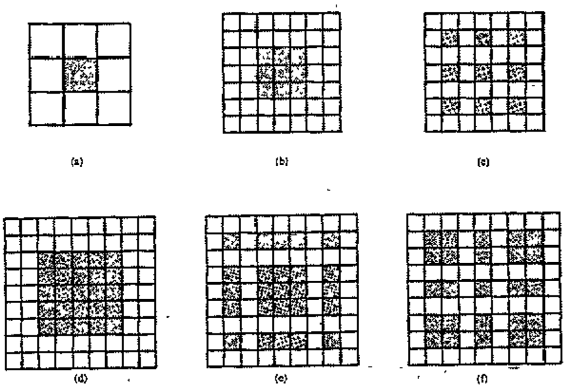

Figure 1. Sierpinski carpets at the first stage of construction (*a*) $b = 3$, $m = 1$; (*b*) and (*c*) $b = 7$, $m = 9$; (*d*), (*e*), (*f*) $b = 9$, $m = 25$.

The proposals are all based on the suggestion of Gefen *et al* [1] to measure lacunarity: the fluctuation of mass around each lattice site. In short, in these studies the lacunarity is calculated from a simple covering method: a square cell of size s is placed at various positions i over the fractal and the number of sites or subsquares—the mass of the fractal—$m_i(s)$ inside each one is computed. All the expressions proposed to define lacunarity mathematically involve the fluctuation of mass

$$\sum_i [m_i(s) - \overline{m}(s)]^2 \qquad \text{with} \qquad \overline{m}(s) \equiv \sum_i m_i(s)/n(s) \tag{1}$$

where $n(s)$ is the total number of different cells.

In the first proposal [1] the cells had a fixed length scale (fixed s) and the covering was taken on the first stage of construction of the fractal (the generator). Later, Lin and Yang [3] took the results for different sizes and averaged them for the final value of lacunarity. Lin [4] extended the average for all sizes in the range $1 \leqslant s \leqslant b-1$, while Wu and Hu [5] suggested that $\overline{m}(s)$ in (1) should be taken simply as the fraction of uneliminated area $\bar{m}(s) = s^2b^2 - m/b^2$ in the $s \times s$ covering of the generator. Taguchi [6] proposed an expression for lacunarity $\Lambda^{(k)}(s)$ where the fluctuation of mass inside cells of size s was computed for the fractal at the kth stage of construction and presented results for $k = 2$.

Each proposal meant an improvement in relation to the previous ones concerning the description of the degree of homogeneity of carpets (at least for those in which the degree of homogeneity could be compared by visual inspection of the generators). On the other hand, they do not take into account the true geometric characteristics of the infinite fractal lattice, but only of the lattice at the finite stages of its construction. As already noted by Taguchi [6], $\Lambda^{(k)}(s)$ is strongly dependent on the stage k and for self-similar fractals with rescaling factor b it should only converge for $k \to \infty$ (in the limit of the infinite fractal lattice), when $x \equiv s/b^k \ll 1$.

Lee and Main [7] had also shown that $\Lambda^{(1)}(s)$ is strongly dependent on the distance scale s for some carpets, so that it is meaningless to speak about lacunarity without a specific reference to the scale s.

Recently, Allain and Cloitre [8] suggested a new method to define lacunarity from the mass distribution of the fractal that can be applied to any self-similar fractal. In their proposal, lacunarity is characterized by its scaling properties, that is, by a scaling function

$$\Lambda^{(k)}(s) = f(s/L) \qquad \text{for} \qquad s \ll L \tag{2}$$

where $L \sim b^k$ is the characteristic size of the fractal set (taking the lattice parameter $a = 1$).

In this work, we calculate explicitly $\Lambda^{(k)}(s)$ in the limit $k \to \infty$, i.e., in the true fractal limit for some fractals, using the definition of Allain and Cloitre [8].

Consider the underlying Euclidean lattice on which the fractal is built (in the case of Sierpinski carpets, for instance, the underlying lattice is the square lattice) and cells of radius r which have centres placed on different sites of that lattice (the cell can be any geometrical figure which has a characteristic radius r). For the characteristic size L of the set, the total number of cells is A_sL^E, where A_s is the factor form of the underlying lattice and E is the Euclidean dimension of the underlying lattice, for unit lattice parameter (for the square lattice, $A_s = 1$ and $E = 2$). The distribution of mass in the collection of cells is given by $n(M, r)$, the number of cells with radius r and mass M. The probability that a cell of radius r has mass M is

$$P(M, r) = \frac{n(M, r)}{A_sL^E}. \tag{3}$$

It should be noticed that the true statistical behaviour can be reached only when $r \ll L$, i.e. when the set is large.

The complete characterization of the mass distribution demands the knowledge of the moments of the distribution,

$$Z^{(q)}(r) = \sum_M M^q P(M, r). \tag{4}$$

The lacunarity at scale r is defined as [8]

$$\Lambda(r) = \frac{Z^{(2)}(r)}{[Z^{(1)}(r)]^2}. \tag{5}$$

Note that the relative fluctuation of mass $\Lambda'(r)$ is given by

$$\Lambda'(r) = \frac{Z^{(2)}(r) - [Z^{(1)}(r)]^2}{[Z^{(1)}(r)]^2} = \Lambda(r) - 1 \tag{6}$$

and that, for translationally invariant lattices, $Z^{(2)}(r) = [Z^{(1)}(r)]^2$, leading to $\Lambda(r) = 1$ (or $\Lambda'(r) = 0$).

The moments (4) follow scaling relations. Consider the first moment of the mass distribution:

$$Z^{(1)}(r) = \bar{M} = \sum_M M\, P(M, r) = \sum_M \frac{M\, n(M, r)}{A_s L^E}. \tag{7}$$

As the centre of the cells goes through all sites of the underlying lattice, each site belonging to the fractal (active site) is counted V times, where $V = A_c r^E$ is the volume of the cell, and A_c is the factor form of the cell. If M_0 is the total mass of the fractal, we have

$$Z^{(1)}(r) = \frac{M_0 A_c\, r^E}{A_s L^E}. \tag{8a}$$

Equation (8a) can be rewritten as

$$Z^{(1)}(r) = z_1 (r/L)^E \tag{8b}$$

with $z_1 \equiv M_0 A_c / A_s$.

Now consider the second moment of the mass distribution:

$$Z^{(2)}(r) = \sum_M \frac{M^2 n(M, r)}{A_s L^E} = M_0^2 \sum_M \frac{(M/M_0)^2 n(M, r)}{A_s L^E}. \tag{9a}$$

As in (8), each site contributes V times. It can be shown [9] that

$$Z_2(r) = M_0^2 \frac{V}{A_s L^E} C(r) \tag{9b}$$

where $C(r)$ scales as $f_0 (r/L)^{D_F}$. Then,

$$Z^{(2)}(r) \simeq M_0^2 \frac{A_c f_0}{A_s} \frac{r^{E+D_F}}{L^{E+D_F}}. \tag{10a}$$

Equation (10a) can also be rewritten as

$$Z^{(2)}(r) \simeq z_2\left(\frac{r}{L}\right)^{E+D_F} \tag{10b}$$

where the prefactor $z_2 \simeq M_0^2 A_c f_0/A_s$, for $r \to \infty$.

Finally, from (5), (8) and (10) lacunarity $\Lambda(r)$ varies according to the power law:

$$\Lambda(r) \simeq \lambda\left(\frac{r}{L}\right)^{D_f-E}. \tag{11}$$

The prefactor λ in (11) characterizes the lacunarity $\Lambda(r)$ of the fractal set and is called the lacunarity parameter [8]. Note that $\lambda \equiv z_2/z_1^2 \simeq f_0 A_s/A_c$ for $1 \ll r \ll L$.

Our calculation of $\Lambda(r)$ is based on a method of graph counting suitable for self-similar regular fractals [10]. The method shows that the number of embeddings of a graph on any stage of construction of the set can be obtained from the calculation of its embeddings on previous stages, due to the fixed rule of construction. For Sierpinski carpets with parameters b and m, it was shown that the number of embeddings $G(k)$ of a particular graph on a carpet at the k-stage varies as

$$G(k) = A(b^2-m)^k + Bb^k + C. \tag{12}$$

Equation (12) also applies for other self-similar regular fractals embedded in two-dimensional Euclidean lattices (for instance, for the Sierpinski gasket [2], with parameters $b = 2$ and $m = 1$).

To find A, B and C, it is sufficient to calculate directly $G(k)$ for three different stages of lattice construction in which the graph can be embedded. From (12), in the limit of the infinite fractal lattice ($k \to \infty$), the number of embeddings behaves as

$$G(k) \simeq A(b^2-m)^k = AL^{D_f} \tag{13}$$

where $D_f = \ln(b^2-m)/\ln b$ is the fractal dimension [2], and $L = b^k$.

As we vary the characteristic lengh L, the mass M_0 of the fractal scales as $M_0 = A_F L^{D_f}$, with A_F calculated from (13) for the number of sites (number of one-site graphs).

From (8), considering the two-dimensional underlying lattice:

$$Z^{(1)}(r) = A_F A_c/A_s \frac{r^2}{L^{2-D_f}}. \tag{14}$$

In order to calculate $Z^{(2)}(r)$, we consider from (9) $Z^{(2)}(r)\, A_s L^2$, the sum of the mass squared within each cell. As we 'glide' the cell through the underlying lattice, each one contains a set of active and non-active sites that forms a connected graph (see figure 2). Each type of graph has a number of embeddings that scales as (12). Its contribution to $Z^{(2)}(r)\, A_s L^2$ at stage k is given by $G(k)M^2$, where M is the number of active sites of the graph. Adding the contributions of all graphs, the result also scales like (12). Again the coefficients A, B and C are obtained from direct calculation of $Z^{(2)}(r)\, A_s L^2$ for three different stages of lattice construction and vary according to the cell radius:

$$Z^{(2)}(r)\, A_s L^2 = A(r)(b^2-m)^k + B(r)\, b^k + C(r). \tag{15a}$$

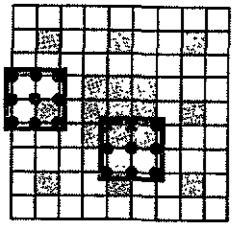

Figure 2. Examples of two possible positions of 2×2 cells ($r = 1$) at the second stage of carpet 1. The active sites within each cell are represented by full circles and non-active sites by crosses.

Table 1. For carpets 1 to 6, corresponding to generators 1(a) to 1(f), the table shows the fractal dimension D_f, the parameters A_F, A_1 and the lacunarity parameter λ (see text).

Carpet	D_f	A_F	A_1	λ
1	1.8928	44/35	21.1 ± 0.8	0.86 ± 0.03
2	1.8957	496/429	18.8 ± 0.8	0.88 ± 0.04
3	1.8957	188/143	21.5 ± 0.9	0.78 ± 0.03
4	1.8320	3088/2585	18.5 ± 1.3	0.81 ± 0.06
5	1.8320	3812/2585	24.3 ± 0.2	0.70 ± 0.01
6	1.8320	3812/2585	24.5 ± 0.6	0.70 ± 0.02

In the limit of an infinite fractal lattice ($k \to \infty$), analogously to (13), (15a) leads to

$$Z^{(2)}(r)\, A_s L^2 = A(r)\ L^{D_f}. \tag{15b}$$

But from (10) and (15b) we expect that

$$A(r) \simeq A_1 r^{2+D_f}. \tag{16}$$

Using (5), (14) and (15b),

$$\Lambda(r) \simeq \frac{A_s A(r)}{(A_c A_F)^2 r^4}\ L^{2-D_f}. \tag{17}$$

From (11), (16) and (17), we obtain that in the true fractal limit ($L \to \infty$),

$$\lambda \simeq \frac{A_s A_1}{(A_c A_F)^2}. \tag{18}$$

Note that our method gives the exact value of $A(r)$ for each r, and that (15b) is exact in the limit $L \to \infty$. There are computational limitations, however, in obtaining the coefficients $A(r)$, $B(r)$ and $C(r)$ in (15a) for arbitrarily large r because this would need calculations on three lattice stages large enough to embed graphs of 'size $2r$'.

By assuming that the asymptotic form (16) holds for $A(r)$, we get (18) with A_1 being the only approximated parameter.

In fact, the asymptotic form of $Z^2(r)\, A_s L^{2-D_f}$ in (15b) for large r is obtained assuming that for r up to $r_{\max}$ there are finite-size corrections in expression (16) for $A(r)$ of the form

$$A(r) \simeq A_1 r^{D_f+2}(1 + B_1/r). \tag{19}$$

Then, from (15b),

$$\frac{Z_2(r)\ A_s L^{2-D_f}}{r^{D_f+1}} \simeq A_1(r + B_1) \tag{20}$$

where the parameters A_1 and B_1 are determined by least-squares fit.

In this work we present results for lacunarity in the true fractal limit for Sierpinski carpets whose generators are shown in figure 1 (the carpets are labelled 1 to 6, respectively). The prefactors A_F in (14) for each lattice are shown in table 1. For them we considered square $2r \times 2r$ cells ($A_c = 4$) and the underlying square lattice ($A_s = 1$).

We have computed explicitily $Z^{(2)}(r)\,L^2$ in the carpet limit (15*b*) for $r \leqslant r_{\max}$. For carpet 1, $r_{\max} = 100$, for carpets 2 and 3, $r_{\max} = 90$ and for carpets 4, 5 and 6, $r_{\max} = 40$. These limits are determined basically by the computing time and memory.

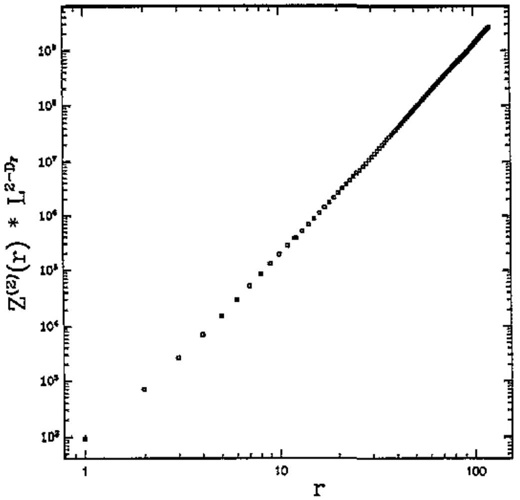

Figure 3. Plot of $\ln[Z^{(2)}(r)\ L^{2-D_f}]$ against $\ln r$ for carpet 1 characterized by parameters $b = 3$ and $m = 1$ (see text).

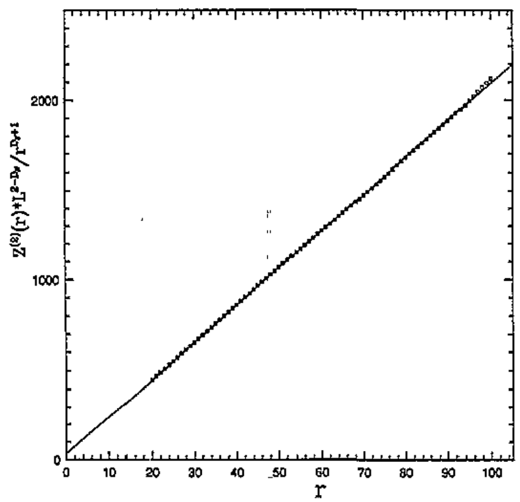

Figure 4. Plot of $Z^2(r)\,L^{2-D_f}/r^{D_f+1}$ against r for carpet 1 for $20 \leqslant r \leqslant 100$.

In figure 3 we plot $\ln[Z^{(2)}(r)\ L^{2-D_f}]$ (which is exact in the limit $L \to \infty$) against $\ln r$ for carpet 1 and in figure 4 we plot $Z^{(2)}(r) \cdot L^{2-D_f}/r^{D_f+1}$ against r for the same carpet. The linear fit (20) gives $A_1 = 20.60 \pm 0.02$ for $20 \leqslant r \leqslant 100$ (this error bar was obtained using standard methods). The quality of the fit (see figure 4) indicates that finite-size corrections are well accounted for by equation (20). The accuracy of this result was tested by estimating A_1 considering various ranges of r, as for instance $20 \leqslant r \leqslant 40$ and $80 \leqslant r \leqslant 100$. The difference between the estimates never exceeds 6%. Our final estimate of A_1 has an error bar that includes the estimations of A_1 using different ranges of r for fitting (20). The same procedure was used for the other carpets.

In table 1 we show the estimated value of A_1 for carpets 1 to 6 as well as the estimated values of the lacunarity parameter λ (18). As lacunarity $\Lambda(r)$ is scale-dependent, λ is the useful parameter to order the carpets according to their degree of inhomogeneity.

Figures 5 and 6 show the behaviour of $\Lambda(r)\ L^{D_f-2}$ (obtained from (17) with the exact value of A_F and the estimation of A_1) for carpets 2 to 6. In each figure, the carpets have the same dimensionality D_f. From (11), $\Lambda(r)\ L^{D_f-2} \simeq \lambda r^{D_f-2}$. This behaviour is confirmed by figures 5 and 6, where a slow decay of $\Lambda(r)\, L^{2-D_f}$ with r is obtained ($2 - D_f = 0.104\ldots$ for carpets of figure 5, and $2 - D_f = 0.168\ldots$ for carpets of figure 6). From figure 5, we conclude that lacunarity $\Lambda(r)$ of carpet 2 is higher than that of carpet 3 no matter how long the scale distance r is. In figure 6 the plot for carpets 4–6 shows that lacunarity $\Lambda(r)$ of carpet 4 is the highest, but we can not resolve carpets 5 and 6 for any scale r.

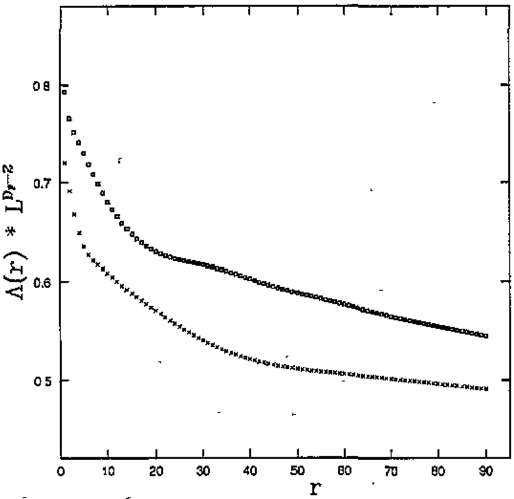

Figure 5. Plot of $\Lambda(r)\ L^{D_f-2}$ against r for carpet 2 (□) and carpet 3 (×) for $r \leqslant 90$.

Our results agree qualitatively with the previous results concerning the ordering of carpets according to their degree of inhomogeneity, but for the first time they are obtained from a calculation of mass distribution for the true fractals and not at finite stages of their construction. For this reason, the coincidence might be accidental. Consequentely, all previous analysis concerning lacunarity of carpets as a possible classifying geometrical parameter for universality or non-universality should be treated with care.

To illustrate the applicability of the method to regular fractals other than carpets, we also calculated the lacunarity in the true fractal limit for the Sierpinski gasket with parameters $b = 2$ and $m = 1$ (see figure 7(a)). For this fractal we considered circular cells

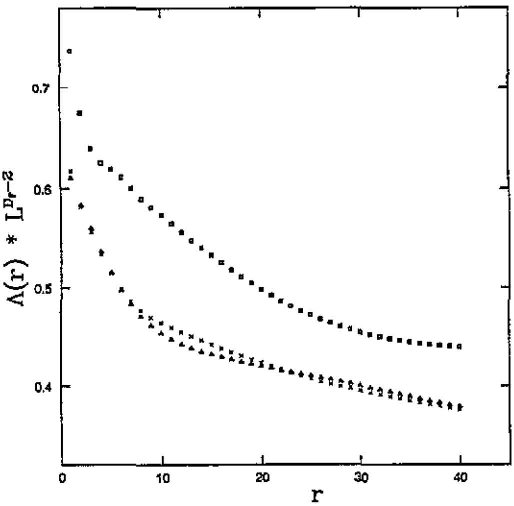

Figure 6. Plot of $\Lambda(r)\ L^{D_f-2}$ against r for carpet 4 (□), carpet 5 (×) and carpet 6 (△) for $r \leqslant 40$.

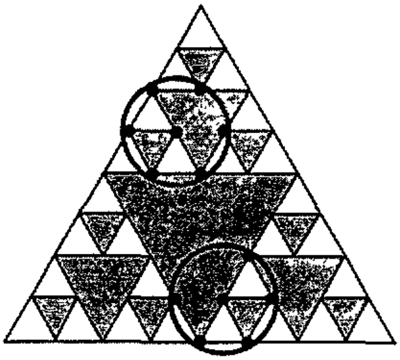

Figure 7. The Sierpinski gasket, characterized by parameters $b = 2$ and $m = 1$, at the second stage of construction and cells ($r = 1$) at two possible positions. The active sites within each cell are represented by full circles.

($A_c = \pi/\sqrt{3/2}$) and the underlying triangular lattice ($A_s = \frac{1}{2}$) in (14), (15), (18) and (20). Similarly, parameters A_1 and B_1 in (20) are determined by least-squares fit.

Figure 8 shows the plot of $\frac{1}{2}\ Z_2(r)L^{2-D_f}/r^{D_f+1}$ against r and the estimated parameter $A_1 = 17.07 \pm 0.04$. Using (18) and the same procedure to obtain the final estimate for A_1, the lacunarity parameter is $\lambda = 0.29 \pm 0.01$. This is the first estimation of lacunarity for the Sierpinski gasket in the literature.

As shown in the last example, the definition of lacunarity used here [8] and our method of calculation [10] may be applied to any self-similar fractal, while previous methods used for carpets are not applicable. Now, it is possible to conduct a fairly general analysis of the relevance of lacunarity for the universality classes of phase transitions on fractals. Work along this line is in progress.

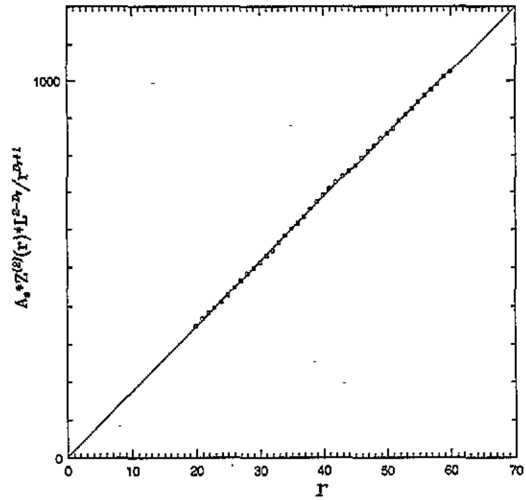

Figure 8. Plot of $A_s Z^{(2)}(r)L^{2-D_f}/r^{D_f+1}$ against r for the Sierpinski gasket.

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